A NOTE ON TOPOLOGICAL DEGREE THEORY FOR HOLOMORPHIC MAPS†

BY

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ABSTRACT

Using degree theory, an elementary topological proof is given of some well-known results in the theory of several complex variables. In particular it is shown that a compact analytic variety consists of finitely many points.

Let $\Omega \subset\subset A \subset \mathbb{C}^n$ where Ω and A are open. Suppose $f: A \to \mathbb{C}^n$ is holomorphic and $b \in \mathbb{C}^n$ with $b \notin f(\partial\Omega)$ (where $\partial\Omega$ denotes the boundary of Ω). Choosing a basis for $\mathbb{C}^n$, a basis for $\mathbb{R}^{2n}$ can be obtained from it in a natural fashion. This identification induces an isomorphism between Ω and a bounded open set $\Lambda \subset \mathbb{R}^{2n}$, between f and a mapping ϕ continuous from $\bar{\Lambda}$ to $\mathbb{R}^{2n}$, and between b and $\beta \in \mathbb{R}^{2n}$ with $\beta \notin \phi(\partial\Lambda)$. The Brouwer degree of the map ϕ relative to the set Λ and the point β is therefore defined; it is denoted in this paper by $d(\phi, \Lambda, \beta)$. This function is integer valued where defined. See, for example, [9, Chap. 3] or [5] for the definition of Brouwer degree and an elementary analytical development of its properties. The degree of f relative to the set Ω and the point b is then defined as equal to $d(\phi,\Lambda,\beta)$ and is also denoted by $d(f,\Omega,b)$.

The properties of topological degree for complex analytic mappings in infinite-dimensional Banach spaces as well as $\mathbb{C}^n$ have been studied by Cronin [3] and Schwartz [8]. A more general class of mappings of analytic type has been treated by Browder [1]. These authors establish in particular (e.g., [1, Th. 1] or [4, Th. 3]).

LEMMA 1. *Let f, Ω, b be as above. Then*

† This research was supported in part by the Office of Naval Research under contract N00014-67-A-0128-0023. Reproduction in whole or in part is permitted for any purpose of the U. S. Government.

Received August 23, 1973

(i) $d(f, \Omega, b) \geqq 0$,

(ii) $d(f, \Omega, b) > 0$ *if and only if* $b \in f(\Omega)$.

Thus the topological degree for holomorphic maps is non-negative and in fact is positive if $b \in f(\Omega)$. To prove Lemma 1, only elementary facts about analytic functions are required, in particular:

(i) $J_\phi(x) = |J_f(z)|^2 \geqq 0$ where $J_f(z)$ denotes the Jacobian determinant of f evaluated at $z \in \mathbb{C}^n$ and $J_\phi(x)$ is the Jacobian determinant of ϕ evaluated at the corresponding $x \in \mathbb{R}^{2n}$;

(ii) $J_{f(z)+\varepsilon z}(z) \neq 0$ for all $0 \neq \varepsilon$ sufficiently small.

For the further development of the theory of degree in [8] and [1], use is made of a well-known result on analytic varieties (e.g., [7, Th. 7] or [4, Chap. 3, Cor. B17]) which states that a compact analytic variety consists of finitely many points. Due to the way in which the argument in [8] is arranged, [7, Th. 7] is already used in the proof of Lemma 1.

The purpose of this note is to show that Lemma 1 can be employed to prove the above result on analytic varieties in a simple topological fashion. In addition, an elementary proof will be given of the fact that $d(f, \Omega, b) = 1$ if and only if there is a unique $\zeta \in \Omega$ such that $f(\zeta) = b$ and $J_f(\zeta) \neq 0$. This has already been shown by Schwartz [8], but his argument is based on a result of Cronin [3] which uses some deep properties of homogeneous polynomials. Our argument by-passes the need for such powerful machinery.

The author acknowledges with thanks helpful conversations with Bent Birkeland, Henrik Stetkaer, and in particular Andrew Browder.

We begin with the following improvement of Part (ii) of Lemma 1.

THEOREM 2. *Let* $f: A \to \mathbb{C}^n$ *where* $A \subset \mathbb{C}^n$ *is open and* f *is holomorphic. Suppose* $\Omega \subset\subset A$ *is open,* $b \in \mathbb{C}^n$, *and* $b \notin f(\partial\Omega)$. *If* $d(f, \Omega, b) = k$, *then* $f^{-1}(b) \cap \Omega$ *contains at most* k *distinct points.*

PROOF. The proof is by induction on k. Suppose $k = 0$. Then by (ii) of Lemma 1, $f^{-1}(b) \cap \Omega = \emptyset$ and the result is trivially true. Next suppose the theorem has been established for $k - 1$ and $d(f, \Omega, b) = k > 0$. If the result is not true, we can find $k + 1$ distinct points $\zeta_0, \cdots, \zeta_k$ in $f^{-1}(b) \cap \Omega$. We can assume $b = 0$ and $\zeta_0 = 0$. The degree is unaffected by the choice of a basis for $\mathbb{C}^n$ [9, Prop. 3.32] and it is not difficult to see that by an appropriate such choice all components ζ_{jm} of ζ_j can be made nonzero, $1 \leqq m \leqq n$, $1 \leqq j \leqq k$. Define a new analytic function $g(z)$ by

$$g_m(z) = f_m(z) + \varepsilon \prod_{j=0}^{k} (z_m - \zeta_{jm}) \quad \text{for } 1 \leqq m \leqq n.$$

Then $g(\zeta_j) = 0$, $0 \leqq j \leqq k$. For ε sufficiently small, $d(f,\Omega,0) = d(g,\Omega,0)$ [9, Th. 3.16 (3)]. Moreover $J_g(0)$ is a polynomial in ε whose leading coefficient is

$$(-1)^{kn} \prod_{m=1}^{n} \prod_{j=1}^{k} \zeta_{jm} \neq 0.$$

Thus we can choose ε near zero so that $J_g(0) \neq 0$. Therefore $z = 0$ is an isolated zero of g and by the additive property of degree [9, Th. 3.16 (5)], $d(g,\Omega,0) = d(g,B_\delta(0),0) + d(g,\Omega - \overline{B_\delta(0)},0)$ where

$$B_\delta(w) = \left\{ z \in \mathbb{C}^n \,\middle|\, |z - w| \equiv \left(\sum_{m=1}^{n} |z_j - w_j|^2 \right)^{\frac{1}{2}} < \delta \right\}$$

and $0 < \delta$ is chosen small enough so that $B_\delta(0) \subset \Omega$ and 0 is the unique zero of g in $\overline{B_\delta(0)}$. Hence by (ii) of Lemma 1, $k \geqq 1 + d(g,\Omega - \overline{B_\delta(0)},0)$ or $d(g,\Omega - \overline{B_\delta(0)},0) = \rho \leqq k - 1$. By the induction hypothesis, g has at most $k-1$ distinct zeroes in $\Omega - \overline{B_\delta(0)}$. But $g(\zeta_j) = 0$, $1 \leqq j \leqq k$ with the ζ_j distinct. Thus we have a contradiction and the proof is complete.

Corollary 2.1. *Let f,Ω,b be as in Theorem 2. If $d(f,\Omega,b) = 1$, there is a unique $\zeta \in \Omega$ such that $f(\zeta) = b$.*

Proof. Immediate from (ii) of Lemma 1 and Theorem 2.

Next we obtain the result on analytic varieties.

Corollary 2.2. *Let f, A, b be as in Theorem 2. If $f^{-1}(b)$ is compact in A, then $f^{-1}(b)$ consists of finitely many points.*

Proof. Since $f^{-1}(b)$ is compact in A, there exists a bounded open set $\Omega \subset \subset A$ with $f^{-1}(b) \subset \Omega$. Hence $d(f,\Omega,b)$ is defined and equals, e.g., k. By Theorem 2, $f^{-1}(b)$ consists of at most k points.

Corollary 2.3. *Let f and A be as in Theorem 2. If $f(\zeta) = b$ for some $\zeta \in A$ and $\mathscr{C}$ is the component of $f^{-1}(b)$ to which ζ belongs, then $\mathscr{C} = \{\zeta\}$ or $\mathscr{C}$ meets every neighborhood of ∂A (that is, $\mathscr{C}$ has a nonempty intersection with every neighborhood of ∂A).*

Proof. If not, it is easy to find a bounded open set $\Omega \subset \subset A$ with $\mathscr{C} \subset \Omega$ and $f \neq b$ on $\partial\Omega$. Then by Corollary 2.2 $f^{-1}(b) \cap \Omega$ consists of finitely many points, contradicting the fact that $\zeta \in f^{-1}(b) \cap \Omega$ is not an isolated solution of $f = b$.

REMARK. Actually the result for varieties does not require both the domain and range of f to be subsets of $\mathbb{C}^n$.

COROLLARY 2.4. *Suppose* $h: A \to \mathbb{C}^m$ *where* $A \subset \mathbb{C}^n$ *is open*, $n > m$, *and* h *is holomorphic. If* $c = h(\zeta)$ *for some* $\zeta \in A$, *the component* $\mathscr{C}$ *of* $h^{-1}(c)$ *to which* ζ *belongs meets every neighborhood of* ∂A.

PROOF. We can assume $c = 0 = \zeta$. Define $h_j(z) \equiv 0$, $m + 1 \leqq j \leqq n$ and $\hat{h}(z) = (h_1(z), \cdots, h_n(z))$. Note that $h^{-1}(0_m) = \hat{h}^{-1}(0_n)$ where we have used the subscripts m and n to distinguish between 0 as an element of $\mathbb{C}^m$ and $\mathbb{C}^n$. If the corollary is not true, 0 is an isolated zero of $\hat{h}$. Choose $\delta > 0$ so that 0 is the unique solution of $\hat{h} = 0$ in $\overline{B_\delta(0)} \subset A$. Define $g_j(z) = h_j(z) + \varepsilon z_j$, $1 \leqq j \leqq m$, and $g_j(z) \equiv 0$, $m + 1 \leqq j \leqq n$. For ε sufficiently small, $d(\hat{h}, B_\delta(0), 0) = d(g, B_\delta(0), 0)$ [3, Th. 3.16 (3)]. Hence by Theorem 2, $z = 0$ is an isolated zero of g. Consider the Jacobian determinant of $(\partial g_j(0)/\partial z_k)$, $1 \leqq j,\ k \leqq m$. It is a polynomial in ε with leading coefficient 1 and therefore does not vanish for $0 \neq \varepsilon$ small. By the analytic version of the implicit function theorem, there exist functions $\phi_j(z_{m+1}, \cdots, z_n)$, $1 \leqq j \leqq m$, analytic near $\tilde{z} \equiv (z_{m+1}, \cdots, z_n) = 0 \in \mathbb{C}^{n-m}$ such that $\phi_j(0) = 0$ and $g_j(\phi_1(\tilde{z}), \cdots, \phi_m(\tilde{z}), \tilde{z}) = 0$ near $\tilde{z} = 0$. But then $z = 0$ is not an isolated zero for g and we have a contradiction.

Next a sharper version of Corollary 2.1 will be obtained.

THEOREM. 3. *Let* $f: A \to \mathbb{C}^n$ *where* $A \subset \mathbb{C}^n$ *is open and* f *is holomorphic. Suppose* $\Omega \subset\subset A$ *is open*, $b \in \mathbb{C}^n$, *and* $b \notin f(\partial\Omega)$. *Then* $d(f, \Omega, b) = 1$ *if and only if*

(i) *there exists a unique* $\zeta \in \Omega$ *with* $f(\zeta) = b$ *and*

(ii) $J_f(\zeta) \neq 0$.

PROOF. The sufficiency is an immediate consequence of the definition of degree [9]. Conversely if $d(f, \Omega, b) = 1$, Corollary 2.1 implies (i). We can assume $b = 0 = \zeta$. Suppose that $J_f(0) = 0$. Then the differential of f at 0, $f'(0)$ has a zero eigenvalue. Using [9, Th. 3.16 (3)] again to perturb f slightly, if necessary, we can assume 0 is a simple eigenvalue of $f'(0)$. Choosing an appropriate basis in $\mathbb{C}^n$, f has the form $f(z) = (\hat{f}(z), f_n(z))$ where $\hat{f}(z) = (f_1(z), \cdots, f_{n-1}(z))$, $z = (\hat{z}, z_n)$, $\hat{z} = (z_1, \cdots, z_{n-1})$, and $\hat{f}(z) = L\hat{z} + 0(|z|^2)$, $f_n(z) = 0(|z|^2)$ at $z = 0$ with L a nonsingular $(n-1) \times (n-1)$ matrix. Moreover by a final application of [9, Th. 3.16 (3)], we can assume

$$\frac{\partial^2 f_n(0)}{\partial z_n^2} = 2a \neq 0.$$

Using the usual multi-index notation, for z near 0 for example, for $|z| \leqq \rho$,

$$f_j(z) = \sum_{|\sigma| \geqq 1} f_{j\sigma} z^{\sigma}, \; 1 \leqq j \leqq n.$$

This implies

$$\Big| f_n(z) - \sum_{|\sigma| = 2} f_{n\sigma} z^{\sigma} \Big| \leqq M_1 |z|^3$$

for $|z| \leqq \rho$ where $M_1 > 0$ is a constant. If z also lies in the set $\{|\hat{z}| \leqq \alpha |z_n|\}$,

$$|f_n(z)| \geqq |az_n^2| - \Big| \sum_{|\sigma| = 2} f_{n\sigma} z^{\sigma} - az_n^2 \Big| - M_1 |z|^3 \geqq \frac{|a|}{2} |z_n|^2$$

provided that $0 < \alpha \leqq \alpha_0$ where α_0 is sufficiently small compared to $\min(1, |a|)$ and $\rho \leqq |a|/4M_1$. Similarly $|\hat{f}(z) - L\hat{z}| \leqq M_2 |z|^2$ for $|z| \leqq \rho$. If z also lies in the set $\{|z_n| \leqq M|\hat{z}|\}$,

$$|\hat{f}(z)| \geqq |L\hat{z}| - M_2|z|^2 \geqq M_3|\hat{z}| - M_2|z|^2 \geqq \frac{M_3}{2}|\hat{z}|$$

provided that $\rho \leqq M_3/4(1+M)M_2$. Choosing, for example, $\alpha = \alpha_0/2$ and $M = 2/\alpha_0$, the two sets $\{|\hat{z}| \leqq \alpha |z_n|\}$ and $\{|z_n| \leqq M|\hat{z}|\}$ cover $\mathbb{C}^n$. Note that if

$$|z| \leqq \rho = \min\left(\frac{|a|}{M_1}, \frac{M_3}{4(1+M)M_2}\right),$$

and f is replaced by

$$f_t(z) \equiv \Big(L\hat{z} + t \sum_{|\sigma| \geqq 2} \hat{f}_{\sigma} z^{\sigma},\; az_n^2 + t(f_n(z) - az_n^2)\Big)$$

where $\hat{f}_{\sigma} = (f_{1\sigma}, \cdots, f_{n-1,\sigma})$ the above estimates are uniform for $t \in [0,1]$. Therefore $0 \notin f_t(\partial B_{\rho}(0))$ for all $t \in [0,1]$ and by the homotopy invariance of degree [9, Th. 3.16 (1)], $d(f, \Omega, 0) = d(f, B_{\rho}(0), 0) = d(f_t, B_{\rho}(0), 0)$ for all $t \in [0,1]$. For $t = 0$, $f_t(z) = (L\hat{z}, az_n^2) \equiv (\hat{g}(\hat{z}), g_n(z_n))$ so the first $n-1$ components of f_0 are not linked to the last component. Since $z = 0$ is an isolated zero of f_0, by [9, Th. 3.16 (6), (7)], $d(f_0, B_{\rho}(0), 0) = d((\hat{g}, g_n), B_r \times B_s, (0, 0)) = d(\hat{g}, B_r 0) d(g_n, B_s, 0)$ where $B_r = \{\hat{z} \in \mathbb{C}^{n-1} \mid |\hat{z}| < r\}$ and $B_s = \{z_n \in \mathbb{C} \mid |z_n| < s\}$ and $r, s > 0$ are arbitrary. L is nonsingular; therefore the definition of degree implies $d(\hat{g}, B_r, 0) = 1$. For $0 \neq c \in \mathbb{C}$ near 0, $d(g_n, B_s, 0) = d(g_n, B_s, c)$ [9, Th. 3.16 (4)]. The equation $g_n(z_n) = c \neq 0$ has two distinct solutions in B_s with $g_n' \neq 0$ at the solutions so the definition of degree implies $d(g_n, B_s, c) = 2$. Since $d(f, \Omega, 0) = 1$, we have a contradiction and the proof is complete.

REMARK. Theorem 3 implies the index of an isolated solution ζ of $f=b$ is greater than or equal to 2 if $J_f(\zeta) = 0$. This is a special case of a result of Cronin [3].

An interesting consequence of Theorem 3 is a result developed by G. R. Clements [2]. (See also [6, Chap. 2, Sec. 19, 20].) (Indeed Clements's result can be used to give a shorter proof of Theorem 3. However we preferred to give an independent and elementary topological argument.)

COROLLARY 3.1. *Let f and A be as in Theorem 3. Then f is 1-1 in a neighborhood of $z = \zeta$ if and only if $J_f(\zeta) \neq 0$.*

PROOF. The sufficiency is obvious. Thus suppose that f is 1-1 near $z = \zeta$. By Lemma 1, (ii) and [9, Theorem 3.16 (4)], for δ sufficiently small,

$$0 < d(f, B_\delta(\zeta), f(\zeta)) = d(f, B_\delta(\zeta), b)$$

for all b near $f(\zeta)$. By Sard's theorem [10] the image of the set of points in $B_\delta(\zeta)$ at which J_f vanishes has measure 0. Therefore we can find b near $f(\zeta)$ such that $f^{-1}(b) \cap B_\delta(\zeta) = \{\hat{z}\}$ and $J_f(\hat{z}) \neq 0$. Consequently by the definition of degree, $d(f, B_\delta(\zeta), b) = 1$ and the result now follows from Theorem 3.

Another easy corollary of Theorem 3 is the following improvement of [1, Theorem 4(c)].

COROLLARY 3.2. *Let f, Ω, b be as in Theorem 3 and let K denote the component of $\mathbb{C}^n - f(\partial\Omega)$ to which b belongs. Then $d(f, \Omega, b) = 1$ if and only if f is a bianalytic map of $f^{-1}(K) \cap \Omega$ onto K.*

PROOF. Immediate from Theorem 3 and the fact that $d(f, \Omega, b)$ is constant on components of $\mathbb{C}^n - f(\partial\Omega)$.

To conclude we note that simple proofs of the analogs of our results for mappings of the form $\Phi(u) = u - T(u)$ where T is a compact analytic map of an infinite dimensional space into itself follow by using Schwartz [8] reduction of the infinite dimensional case to that for $\mathbb{C}^n$.

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